Resurgence & Fractal Geometry

Oral Examination Winter 2021

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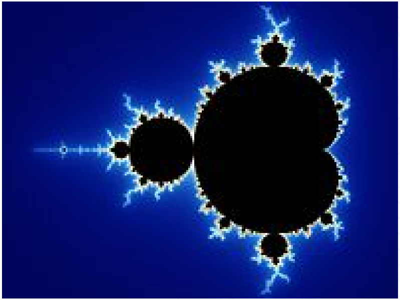
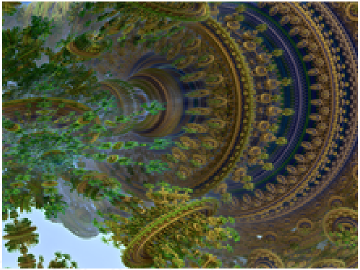
- Fractal Geometry
- Resurgent Asymptotics
- Explicit Formulae

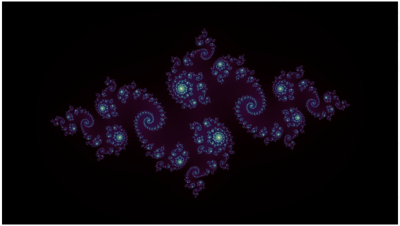
Fractal Geometry

Navigation Shortcuts

Fractals







Fractal Geometry and Geometric Oscillations

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Properties of the fractal can be expressed in terms of these complex dimensions, such as the volume of a neighborhood within a certain distance of the fractal.

Example: The Cantor Set

The standard middle-thirds Cantor set:

At the n^{th} stage, 2^{n-1} intervals of length 3^{-n} are removed.

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The fractal zeta function ζ_{CS} is given by:

$$\zeta_{CS}(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} \left(\frac{1}{3^n}\right)^s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{1}{3^s - 2}$$

Fractal String and Zeta Function

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The zeta function associated to $\mathcal{L}$ is given by:

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s$$

Ordinary Fractal String as a Measure

An ordinary fractal string $\mathcal{L} = \{\ell_n\}_{n\in\mathbb{N}}$ may be represented as a measure: [1]

$$\mu_{\mathcal{L}} = \sum_{j=1}^{\infty} \delta_{\{\ell_j^{-1}\}}$$

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This construction works for any sufficiently nice measure, not just those from fractal strings.

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Generalized Fractal String

Definition

A **generalized fractal string** is a local positive or complex measure η defined on $(0, \infty)$.[2] We also stipulate that η has no mass near zero, i.e. there exists a positive number x_0 for which $|\eta|[(0, x_0)] = 0$, where $|\eta|$ denotes the variation of η.

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More on the Counting Function

Ordinary Counting Function

The **geometric counting function** of an ordinary fractal string $\mathcal{L}$:

$$N_{\mathcal{L}}(x) := \int_0^x d\mu_{\mathcal{L}} = \sum_{\ell_n^{-1} \leq x} 1$$

counts the number of reciprocal lengths up to the input.*

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counts the number of reciprocal lengths up to the input.*

*By convention, the counting function at jump discontinuities is defined to be the average of the lateral limits.

For a general measure η, we write:

$$N_\eta(x) = \int_0^x d\eta = \eta\big((0,x)\big) + \frac{1}{2}\eta(\{x\})$$

Zeta Functions and Complex Dimensions

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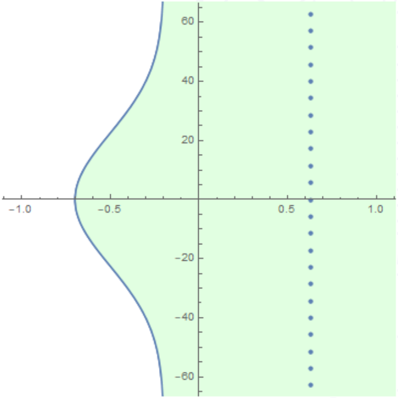
Golden String:

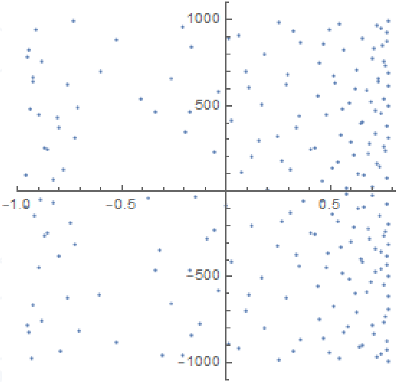
$$\zeta_{GS}(\omega) = \frac{1}{1 - 2^{-\omega} - 2^{-\varphi\omega}} = \infty \iff 2^{-\omega} + 2^{-\varphi\omega} = 1$$

Complex Dimensions Plotted

The Cantor String (Screened)



The Golden String



Explicit Fomulae

Navigation Shortcuts

Namesake: Riemann's Explicit Formula

Let $f(x)$ denote the prime power counting function, and $\zeta(s)$ the Riemann zeta function. In particular:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \pi(x^{1/n})$$

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Riemann wrote the formula (proved later by von Mangoldt):

$$f(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + \int_x^{\infty} \frac{1}{x^2 - 1} \frac{dx}{x \log x} - \log 2$$

where the sum is taken over critical zeroes, in order of increasing imaginary part magnitude. [3]

[3] See [Edw74] for more detail.

Explicit Formula via Complex Dimensions

Pointwise E.F., with Error (Thm 5.10 in [LvF13])

Let η be a *languid* generalized fractal string, k a sufficiently large positive integer, [4] and $D_\eta(W)$ the visible complex fractal dimensions of η in the window W to the right of screen S. Then for all $x > 0$,

$$
\begin{aligned}
N_\eta^{[k]}(x) &= \sum_{\omega \in D_\eta(W)} \operatorname{res}\left(\frac{x^{s+k-1}\zeta_\eta(s)}{(s)_k}; \omega \right) \\
&\quad + \frac{1}{(k-1)!} \sum_{\substack{j=0 \\ -j \in W \setminus D_\eta}}^{k-1} \binom{k-1}{j} (-1)^j x^{k-1-j} \zeta_\eta(-j) \\
&\quad + O\left(x^{\sup \operatorname{Re}(S)+k-1} \right)
\end{aligned}
$$

[4] Specifically, $k > \max\{1, \kappa + 1\}$, where κ is from the languid growth conditions to be defined on the next slide.

Explicit Formula Notes

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- These formulae can be established for any k when considered in the distributional sense.

- Explicit formulae can also been established for other functions such as geometric tube functions.

Resurgent Asymptotics

Navigation Shortcuts

Asymptotic Expansions

We say $f(z) \sim \sum_{n=1}^{\infty} a_n \frac{1}{z^n}$ as $z \to \infty$ provided that each partial sum truncation is an approximation to f with error on the order of the next term in the series.

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Equivalent definitions: As $z \to \infty$,

$$f(z) \sim \sum_{n=1}^{\infty} a_n \frac{1}{z^n}$$

$$f(z) = \sum_{n=1}^{N} a_n \frac{1}{z^n} + O\left(\frac{1}{z^{N+1}}\right)$$

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Asymptotic Expansion Examples

Stirling's series:

$$\log(\Gamma(x)) \sim \left(x - \tfrac{1}{2}\right)\log(x) - x + \frac{1}{2}\log(2\pi) + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)} x^{-2j+1}, \quad x \to \infty$$

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Sine & non-uniqueness

$$\sin(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \sim \sin(z) + e^{-1/z}, \quad z \to 0^+$$

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Non-example (a simple transseries)

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} + e^{-x}, \quad x \to +\infty$$

Stokes Phenomenon

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Transseries are a broader class of series that can contain all of the important terms. We make sense of them via stronger Borel resummation techniques.

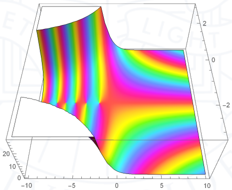
Supernumerary Bows & The Airy Function

Airy Function & Stokes Phenomenon

The Airy function has two different asymptotic expansions.
To first order:

$$\mathrm{Ai}(z)$$

(Entire)



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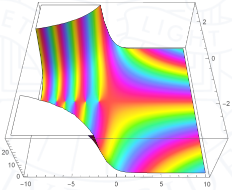
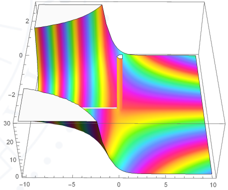
(Entire)	$\lvert\arg(z)\rvert < \pi$





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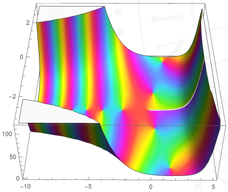
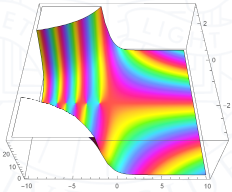
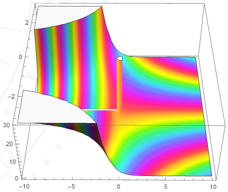
$\frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}}\sin\left(\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right)$	$\mathrm{Ai}(z)$	$\frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}z^{\frac{3}{2}}}$
$\lvert\arg(-z)\rvert<\frac{2\pi}{3}$	(Entire)	$\lvert\arg(z)\rvert<\pi$







Airy Function Expansion

The Airy function is governed by the asymptotic expansion:

$$\varphi_{\mathrm{Ai}}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\Gamma(n+1)} \frac{1}{z^n}$$

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More remarks:

- φ_{Ai} is factorially divergent.
- $z = k^{\frac{3}{2}}$ is a natural change of variables for ensuing resummation.

Borel Summation

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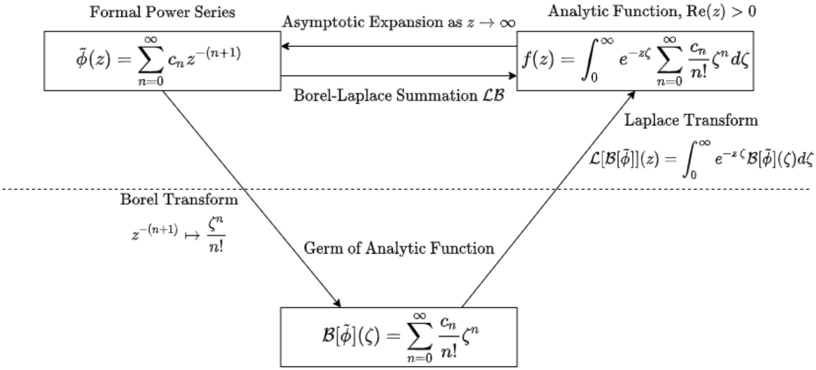
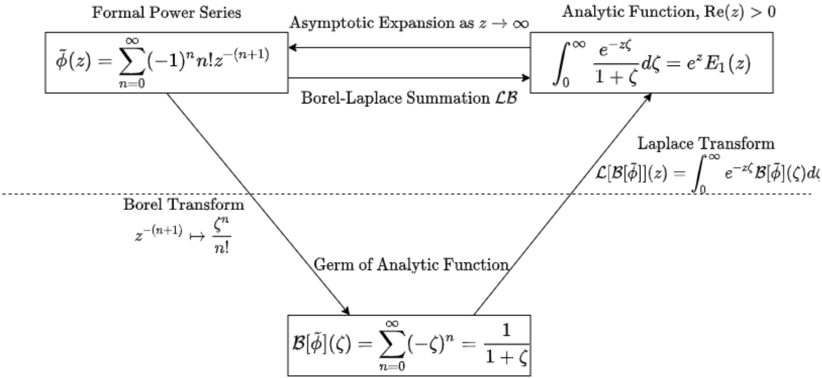
Borel Summation: Schematic



Borel Summation: Example



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For example, if we chose $\tilde{\varphi}(z) = \sum_{n=0}^{\infty} n! z^{-(n+1)}$, its Borel transform would have a a singularity at $+1$, preventing an ordinary Laplace transform.

Airy Series: Borel Summation

- The minor of φ_{Ai} is its (formal) Borel transform, forgetting the constant term:

$$\tilde{\varphi}_{\text{Ai}} := \mathcal{B}[\varphi_{\text{Ai}}] = \sum_{n=1}^{\infty} a_n \frac{\zeta^{n-1}}{(n-1)!}$$

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- $\tilde{\varphi}_{\mathrm{Ai}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \{0, -\frac{4}{3}\}$
- For any direction θ not along the negative real axis, the following converges for $\mathrm{Re}(ze^{i\theta}) > 0$:

$$S_\theta \varphi_{\mathrm{Ai}}(z) := a_0 + \mathcal{L}_\theta \mathcal{B}[\varphi_{\mathrm{Ai}}](z) = a_0 + \int_0^{\infty e^{i\theta}} \tilde{\varphi}_{\mathrm{Ai}}(\zeta) e^{-z\zeta} d\zeta$$

A Borel Resummed Expansion

Where before:

$$\mathrm{Ai}(k) \sim \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

We now have:

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One can rotate the direction of summation for new regions of validity.

Transseries Short Introduction

A transseries starts as a formal asymptotic series, but allowing for multiple levels of exponentials and logarithms. They arise as a closure of power series under algebraic and differential operations.

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These (summable) transseries are in correspondence with analytic germs of so-called *analyzable* functions. These functions are, loosely speaking, Borel transforms of at-most-factorially divergent asymptotic expansions which can be analytically continued in the Borel plane.

Transseries & Analyzability

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Formal Power Series} \arrow[d] & \text{Summable Power Series} \arrow[l, "\text{Embedding}"'] \arrow[r, shift left, "\text{Resummation}"] \arrow[d, "\text{Closure under } (+,\times,\circ,\partial) \text{ and } (+,\times,\circ,\partial)^{-1}"] & \text{Analytic Germs} \arrow[l, shift left] \arrow[d] \\
\text{Formal Transseries} & \text{Summable Transseries} \arrow[l] \arrow[r, shift left] & \text{Analyzable Germs} \arrow[l, shift left, "\text{Formalization}"]
\end{tikzcd}
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Resurgent Functions

(Provisional) Definition

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These functions form an algebra with addition and multiplication (the latter becoming convolution in the Borel plane.)

Behavior of Singularities

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When there are singularities, an extra Hankel contour can be introduced to connect integrals along a ray above or below the singularity. The machinery of resurgent asymptotics involves an operator that relates the behavior of these two contours. This so-called alien derivative connects the behavior of the analytic germ near the origin to the behavior near other singular points.

Airy Function Resummation along $\mathbb{R}^-$

Depiction from [Del06]:

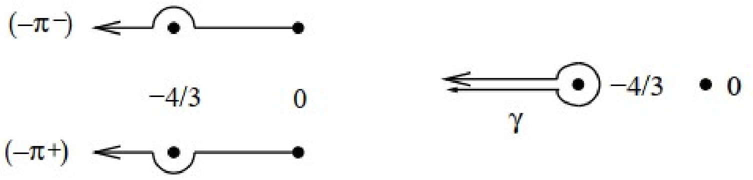

FIGURE 2. Right and left Borel-resummation.

One can compares right and left-resummations, since

$$\mathrm{S}_{-\pi^-}\varphi_{Ai}(z) = \mathrm{S}_{-\pi^+}\varphi_{Ai}(z) + \int_\gamma \widetilde{\varphi_{Ai}}(\zeta)e^{-z\zeta}\,d\zeta \tag{4}$$

Alien Calculus & Behavior across the Singularity

The Hankel contour γ can be expressed using the so-called alien derivative:

$$\int_\gamma \tilde{\varphi}_{\text{Ai}}(\zeta)e^{-z\zeta}d\zeta = e^{+\frac{4}{3}z}S_{-\pi}\left(\Delta^z_{-\frac{4}{3}}\varphi_{\text{Ai}}\right)(z)$$

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In this case,

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φ_{Bi} is also Gevrey-1 and its minor $\tilde{\varphi}_{\mathrm{Bi}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \{0, +\frac{4}{3}\}$.

More on the Airy Function.

Namesake: Resurgence

Écalle on coining “Resurgence”

[Alien derivatives] enable us to describe, by means of so-called resurgence equations of the form $E_\omega(\overset{\triangledown}{\phi}, \Delta_\omega \overset{\triangledown}{\phi}) \equiv 0$, the very close connection which usually exists between the behavior of $\hat{\phi}(\zeta)$ near $0_\bullet$ and near its other singular points ω.

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This self-reproduction property is an outstanding feature of all resurgent functions of natural origin (their birth-mark, as it were!) and it is precisely what the label "resurgence" (bestowed somewhat promiscuously on the whole algebra $\mathrm{R\overset{\triangledown}{E}S}$) is meant to convey.

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Scenarios where transseries and Borel resummation have been applied to fractal geometry:

- Transseries formulae have been useful in describing quasi-disk Julia sets.
- Resummation can extend some lacunary Dirichlet series possessing natural boundaries.

Some Observations

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- *Exact* formulae are not expected candidates for extended expansions. On the other hand, divergent expressions, natural boundaries, and other “at worst factorially intractible” behaviors are likely candidates for resurgent properties.
- Discrete measures have piecewise constant counting functions, so we do not expect them to have analytically continuable explicit formulae expansions.

Notable Applications of Resurgent Asymptotics

Dulac's Conjecture

- On finiteness of limit cycles; related to Hilbert's 16^{th} problem
- Écalle's proof relies on resurgent functions

Quantum Field Theory

- Exponentially small, non-analytic corrections to perturbative expansions ("instantons")
- Potential to recovering nonperturbative effects through resurgence of a perturbative expansion

More Applications in Mathematical Physics

- Normal forms of dynamical systems
- Gauge theory of singular connections
- Quantization of symplectic and Poisson manifolds
- Floer homology and Fukaya categories
- Knot invariants
- Wall-crossing and stability conditions in algebraic geometry
- Spectral networks
- WKB approximation in quantum mechanics
- Non-linear differential equations and asymptotics

Explicit Formulae: Proof of the Prime Number Theorem

A Formula for the Riemann Zeta Function

Let ζ be the Riemann zeta function; it is strongly languid with $k = 0$ and $A = 1$. Denote by $\mathcal{P} = \sum_{m\geq 1,p}(\log p)\delta_{\{p_m\}}$ the geometric zeta function of the prime string. Then for all $x > 1$, (in a distributional sense,)

$$\mathcal{P} = 1 - \sum_{\rho} x^{\rho-1} + \sum_{n=1}^{\infty} x^{-(2n+1)}$$

This formula can be used to derive the following formula for the prime counting function π, and thus the prime number theorem.

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-c\sqrt{\log x}})$$

End of Presentation

Thank you for listening!

Appendix: Navigation Shortcuts

Appendix: Languid Growth Conditions

- Let S denote the screen for ζ_η, viz. $S = \{s(t) + it : t \in \mathbb{R}\}$.

Return to pointwise explicit formula with error term.

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Polynomial growth on a sequence of horizontal lines (L1)

$$\forall n \in \mathbb{Z}, \forall \sigma \geq s(T_n), \quad |\zeta_\eta(\sigma + iT_n)| \leq C(|T_n| + 1)^\kappa$$

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$$\forall n \in \mathbb{Z}, \forall \sigma \geq s(T_n), \quad |\zeta_\eta(\sigma + iT_n)| \leq C(|T_n| + 1)^\kappa$$

Polynomial growth along the given screen (L2)

$$\forall t \in \mathbb{R}, |t| \geq 1, \quad |\zeta_\eta(s(t) + it)| \leq |t|^\kappa$$

Return to pointwise explicit formula with error term.

Appendix: Airy Function on $\mathbb{R}^-$

Deducing the behavior Ai for negative real inputs.

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Airy expansion when $|\arg(k) - \pi| < \frac{\pi}{3}$, $z = k^{\frac{3}{2}}$:

$$\mathrm{Ai}(k) = \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} \left(e^{-\frac{2}{3}z} S_{-\frac{3\pi}{2}} \varphi_{\mathrm{Ai}}(z) + i e^{+\frac{2}{3}z} S_{-\frac{3\pi}{2}} \varphi_{\mathrm{Bi}}(z) \right)$$

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Note the new exponential term that appeared.

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Note the new exponential term that appeared.

Once can rewrite the LHS as the resummed version of the second expansion we saw previously.

Return to Airy Resummation.

BibTeX References I

- G. B. Airy, *On the intensity of light in the neighbourhood of a caustic*, Trans. Cambridge Phil. Soc. **6** (1838), 397–402.
- M. V. Berry, *Uniform asymptotic smoothing of Stokes's discontinuities*, Proc. Roy. Soc. London Ser. A **422** (1989), no. 1862, 7–21. MR 990851
- ______, *Waves near Stokes lines*, Proc. Roy. Soc. London Ser. A **427** (1990), no. 1873, 265–280. MR 1039788
- M. V. Berry and C. J. Howls, *Hyperasymptotics for integrals with saddles*, Proc. Roy. Soc. London Ser. A **434** (1991), no. 1892, 657–675. MR 1126872
- ______, *High orders of the Weyl expansion for quantum billiards: resurgence of periodic orbits, and the Stokes phenomenon*, Proc. Roy. Soc. London Ser. A **447** (1994), no. 1931, 527–555. MR 1316622
- Ovidiu Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 141, CRC Press, Boca Raton, FL, 2009. MR 2474083
- Les Cowley, *Bow from 0.75mm diameter drops illuminated by a distant point source*, atoptics.co.uk, 2010, [Simulated Image, Cropped].

BibTeX References II

Eric Delabaere, *Effective resummation methods for an implicit resurgent function*, arXiv:math-ph/0602026, 2006.

Daniele Dorigoni, *An introduction to resurgence, trans-series and alien calculus*, Ann. Physics **409** (2019), 167914, 38. MR 4000056

Jean Écalle, *Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture*, Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 408, Kluwer Acad. Publ., Dordrecht, 1993, pp. 75–184. MR 1258519

H. M. Edwards, *Riemann's zeta function*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974, Pure and Applied Mathematics, Vol. 58. MR 0466039

Geek3, *Airy ai asymptotic*, Wikimedia Commons, 2015, [SVG, Converted to PNG].

Michel L. Lapidus and Machiel van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions*, second ed., Springer Monographs in Mathematics, Springer, New York, 2013, Geometry and spectra of fractal strings. MR 2977849

BibTeX References III

Mika-Pekka Markkanen, *Supernumerary rainbows*, Wikimedia Commons, 2010, [Photograph].

A. B. Olde Daalhuis, S. J. Chapman, J. R. King, J. R. Ockendon, and R. H. Tew, *Stokes phenomenon and matched asymptotic expansions*, SIAM J. Appl. Math. **55** (1995), no. 6, 1469–1483. MR 1358785

Brent Pym, *Resurgence in geometry and physics: Lecture notes*, `https://www.math.mcgill.ca/bpym/courses/resurgence/`, 2016, Accessed: 2021-01-01.

David Sauzin, *Resurgent functions and splitting problems*, 2007.

G. G. Stokes, *On the numerical calculation of a class of definite integrals and infinite series*, Trans. Cambridge Phil. Soc. **9** (1850), 166–187.

———, *On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series*, Acta Math. **26** (1902), no. 1, 393–397, A letter to the editor. MR 1554970